

SU(2) gauge theory of gravity with topological invariants

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Abstract

The most general gravity Lagrangian in four dimensions contains three topological densities, namely Nieh-Yan, Pontryagin and Euler, in addition to the Hilbert-Palatini term. We set up a Hamiltonian formulation based on this Lagrangian. The resulting canonical theory depends on three parameters which are coefficients of these terms and is shown to admit a real $SU(2)$ gauge theoretic interpretation with a set of seven first-class constraints. Thus, in addition to the Newton's constant, the theory of gravity contains three (topological) coupling constants, which might have non-trivial imports in the quantum theory.

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I. INTRODUCTION

The classical dynamics of a system is not affected by the addition of topological densities in the Lagrangian. This is so because such densities can always be locally written as total divergences. However, quantum dynamics might depend on them. The cases of the Sine-Gordon quantum mechanical model or QCD provide perfect examples of such a phenomenon where topological terms leave their imprints on the quantum theory[1].

In gravity theory in 3+1 dimensions, there are three possible topological terms, namely, Nieh-Yan, Pontryagin and Euler, which can be added to the Lagrangian. In terms of tetrads and spin-connections, these can be written as¹:

$$\begin{aligned} I_{NY} &= e \Sigma_{IJ}^{\mu\nu} \tilde{R}_{\mu\nu}{}^{IJ}(\omega) + \epsilon^{\mu\nu\alpha\beta} D_\mu(\omega) e_{I\nu} D_\alpha(\omega) e_\beta^I = \partial_\mu [\epsilon^{\mu\nu\alpha\beta} e_\nu^I D_\alpha(\omega) e_{I\beta}] \\ I_P &= \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega) R_{\alpha\beta}{}^{IJ}(\omega) = 4\partial_\mu \left[\epsilon^{\mu\nu\alpha\beta} \omega_\nu{}^{IJ} \left(\partial_\alpha \omega_{\beta IJ} + \frac{2}{3} \omega_{\alpha I}{}^K \omega_{\beta KJ} \right) \right] \\ I_E &= \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega) \tilde{R}_{\alpha\beta}{}^{IJ}(\omega) = 4\partial_\mu \left[\epsilon^{\mu\nu\alpha\beta} \tilde{\omega}_\nu{}^{IJ} \left(\partial_\alpha \omega_{\beta IJ} + \frac{2}{3} \omega_{\alpha I}{}^K \omega_{\beta KJ} \right) \right] \end{aligned} \quad (1)$$

where $R_{\mu\nu}{}^{IJ}(\omega) = \partial_{[\mu} \omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK} \omega_{\nu]K}{}^J$ and $D_\mu(\omega) e_\nu^I = \partial_\mu e_\nu^I + \omega_\mu{}^{IJ} e_{\nu J}$. Although these topological densities are functions of local geometric quantities, they encode only the global properties of the manifold. The Nieh-Yan density depends on torsion and in Euclidean theory, its integral over a compact manifold is a sum of three integers associated with the homotopy maps $\pi_3(SO(5)) = Z$ and $\pi_3(SO(4)) = Z + Z$. Among the other two which depend on the curvature, the Pontryagin-class characterises the integers corresponding to the map $\pi_3(SO(4)) = Z + Z$ and the Euler-class characterises the combination of Betti numbers. While the first two densities are P and T odd, the third is P and T even (see [2] and the references within).

In order to understand their possible import in the quantum theory, it is important to set up a classical Hamiltonian formulation of the theory containing all these terms in the action. In ref.[3], such an analysis has been presented for a theory based on Lagrangian density containing the standard Hilbert-Palatini term and the Nieh-Yan density. The resulting theory, in time gauge, has been shown to correspond to the well-known canonical gauge theoretic formulation of gravity based on Sen-Ashtekar-Barbero-Immirzi *real* $SU(2)$ gauge fields [4]. Here η^{-1} , the inverse of the coefficient of Nieh-Yan term, is identified with

¹ The quantity \tilde{X}^{IJ} the dual of X_{IJ} in the internal space: $\tilde{X}^{IJ} = \frac{1}{2} \epsilon^{IJKL} X_{KL}$

the Barbero-Immirzi parameter γ . The framework in [3] supersedes the earlier formulation of Holst [5] in the sense that unlike the Holst term, the Nieh-Yan density

(a) does not need any further modifications for the inclusion of matter couplings and the equations of motion continue to be independent of η for all couplings;

(b) provides a topological interpretation for Barbero-Immirzi parameter, leading to a complete analogy between η and the θ -parameter of non-abelian gauge theories (from the classical perspective).

As an elucidation of these facts, the method has been applied to spin- $\frac{1}{2}$ fermions[3] and supergravity theories[6].

Here we include all three topological terms in the Hilbert-Palatini Lagrangian[2]:

$$\mathcal{L}(e, \omega) = \frac{1}{2} e \Sigma_{IJ}^{\mu\nu} R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2} I_{NY} + \frac{\theta}{4} I_P + \frac{\phi}{4} I_E \quad (2)$$

where, $\Sigma_{IJ}^{\mu\nu} = \frac{1}{2} e_{[I}^{\mu} e_{J]}^{\nu}$. In order to understand how the canonical theory of gravity gets affected by such additions, a Hamiltonian analysis based on this Lagrangian is presented below, demonstrating how we obtain a real SU(2) formulation of gravity with all three topological densities.

II. HAMILTONIAN FORMULATION

A convenient way to proceed is to decompose the 16 tetrad fields e_{μ}^I into the fields V_a^I , M_I , N^a and N ($16=9+3+3+1$) (see [2] for further details):

$$\begin{aligned} e_t^I &= N M^I + N^a V_a^I, & e_a^I &= V_a^I; \\ e_I^t &= -\frac{M_I}{N}, & e_I^a &= V_I^a + \frac{N^a M_I}{N}; \\ M_I V_a^I &= 0, & M_I M^I &= -1; \\ V_a^I V_I^b &= \delta_a^b, & V_a^I V_J^a &= \delta_J^I + M^I M_J. \end{aligned} \quad (3)$$

Next, instead of the variables V_I^a and M^I , we define a new set of 12 variables as:

$$E_i^a = 2e \Sigma_{0i}^{ta} \equiv e (e_0^t e_i^a - e_i^t e_0^a) = -\sqrt{q} M_{[0} V_{i]}^a, \quad \chi_i = -M_i/M^0 \quad (4)$$

Before writing the full Lagrangian, we note that with the help of Bianchi identities

$\epsilon^{abc} D_a(\omega) R_{bcIJ} = 0$ and $\epsilon^{abc} D_a(\omega) \tilde{R}_{bcIJ} = 0$, the last two terms in (2) can be written as²:

$$\frac{\theta}{4} I_P + \frac{\phi}{4} I_E = e_{IJ}^a \partial_t \omega_a^{(\eta)IJ} \quad (5)$$

with $(1 + \eta^2) e_{IJ}^a = \epsilon^{abc} \left\{ (\theta + \eta\phi) R_{bcIJ}(\omega) + (\phi - \eta\theta) \tilde{R}_{bcIJ}(\omega) \right\}$. Using (3), (4) and (5), the Lagrangian in (2) can be written as:

$$\mathcal{L} = \pi_{IJ}^a \partial_t \omega_a^{(\eta)IJ} + t_I^a \partial_t V_a^I - NH - N^a H_a - \frac{1}{2} \omega_t^{IJ} G_{IJ} \quad (6)$$

where $\pi_{IJ}^a = e \Sigma_{IJ}^{ta} + e_{IJ}^a$ and

$$\begin{aligned} G_{IJ} &= -2D_a(\omega) \pi_{IJ}^{a(\eta)} - t_{[I}^a V_{J]a} \quad , \\ H_a &= \pi_{IJ}^b R_{ab}^{(\eta)IJ}(\omega) - V_a^I D_b(\omega) t_I^b \quad , \\ H &= \frac{2}{\sqrt{q}} \left(\pi_{IK}^{a(\eta)} - e_{IK}^{a(\eta)} \right) \left(\pi_{JL}^{b(\eta)} - e_{JL}^{b(\eta)} \right) \eta^{KL} R_{ab}^{IJ}(\omega) - M^I D_a(\omega) t_I^a \quad . \end{aligned} \quad (7)$$

Since there are no velocities associated with the fields N , N^a and ω_t^{IJ} , we have the constraints $H \approx 0$, $H_a \approx 0$, $G_{IJ} \approx 0$.

Next, we split the 18 spin-connection fields ω_a^{IJ} as:

$$A_a^i \equiv \omega_a^{(\eta)0i} = \omega_a^{0i} + \eta \tilde{\omega}_a^{0i}, \quad K_a^i \equiv \omega_a^{0i} \quad . \quad (8)$$

The rationale behind such a choice is to make the SU(2) interpretation transparent, as can be understood by noting that A_a^i transforms as connection and K_a^i as adjoint representation under the SU(2) gauge transformations. Also, it is convenient (although not necessary) to work in the time gauge where the boost constraints are solved by the gauge choice $\chi_i = 0$. Thus, in this gauge, the symplectic form becomes:

$$\pi_{IJ}^a \partial_t \omega_a^{(\eta)IJ} + t_I^a \partial_t V_a^I = \hat{E}_i^a \partial_t A_a^i + \hat{F}_i^a \partial_t K_a^i + t_i^a \partial_t V_a^i \quad (9)$$

with

$$\hat{E}_a^i \equiv -\frac{2}{\eta} \tilde{\pi}_{0i}^{a(\eta)} \equiv -\frac{2}{\eta} (\tilde{\pi}_{0i}^a - \eta \pi_{0i}^a) = E_i^a - \frac{2}{\eta} \tilde{e}_{0i}^{a(\eta)}(A, K) \quad (10)$$

$$\hat{F}_i^a \equiv 2 \left(\eta + \frac{1}{\eta} \right) \tilde{\pi}_{0i}^a = 2 \left(\eta + \frac{1}{\eta} \right) \tilde{e}_{0i}^a(A, K) \quad (11)$$

² The quantity $X^{(\eta)IJ}$ is defined as: $X^{(\eta)IJ} = X^{IJ} + \eta \tilde{X}^{IJ}$

Here, the fields V_a^i and its conjugate t_i^a are not independent; they obey the following second-class constraints:

$$V_a^i - \frac{1}{\sqrt{E}} E_a^i \equiv 0, \quad t_i^a - \eta \epsilon^{abc} D_b(\omega) V_c^i = \epsilon^{abc} (\eta D_b(A) V_c^i - \epsilon^{ijk} K_b^j V_c^k) \quad (12)$$

Similarly, eq.(11) shows that the momenta \hat{F}_i^a obey constraints of the form

$$\chi_i^a := \hat{F}_i^a - f(A_b^j, K_c^k) \approx 0 \quad (13)$$

These imply secondary constraints:

$$[\chi_i^a(x), H(y)] \approx 0 \Rightarrow t_i^a - \left(\frac{1 + \eta^2}{\eta^2} \right) \left\{ \eta \epsilon^{ijk} D_b(A) \left(\sqrt{E} E_j^a E_k^b \right) + \sqrt{E} E_j^{[a} E_i^{b]} K_b^j \right\} \approx 0 \quad (14)$$

The solution of (14) can be expressed in the form: $K_a^i - \kappa_a^i(A_b^j, E_k^c) \approx 0$. Since K_a^i and \hat{F}_i^a are canonically conjugate, these constraints evidently form a second-class pair with (13). The constraints (12), (13) and (14), alongwith the constraints $G_i^{rot} \approx 0$, $H_a \approx 0$, $H \approx 0$, completely characterise the canonical theory corresponding to the Lagrangian density (2).

Notice that for $\theta = 0$ and $\phi = 0$, the momenta \hat{F}_i^a in (11) vanishes. This corresponds to the Barbero-Immirzi formulation. Thus, the effect of the addition of Pontryagin and Euler terms in the Lagrangian gets reflected through a richer symplectic structure characterised by a non-vanishing \hat{F}_i^a . Also, for non-vanishing θ and ϕ , the canonical conjugate of the connection A_a^i is \hat{E}_i^a , and not the densitized triad E_i^a as in the case for $\theta = 0$, $\phi = 0$.

III. SU(2) INTERPRETATION

The second-class constraints can all be implemented by using the corresponding Dirac brackets instead of the Poisson brackets. After imposing all the second-class pairs strongly, we are left with a set of seven first class constraints:

$$\begin{aligned} G_i^{rot} &\equiv \eta D_a(A) \hat{E}_i^a + \epsilon^{ijk} K_a^j \hat{F}_k^a \approx 0 \\ H_a &\equiv \hat{E}_i^b F_{ab}^i(A) + \hat{F}_i^b D_{[a}(A) K_{b]}^i - K_a^i D_b(A) \hat{F}_i^b - \eta^{-1} G_i^{rot} K_a^i \approx 0 \\ H &\equiv \frac{\sqrt{E}}{2\eta} \epsilon^{ijk} E_i^a E_j^b F_{ab}^k(A) - \left(\frac{1 + \eta^2}{2\eta^2} \right) \sqrt{E} E_i^a E_j^b K_{[a}^i K_{b]}^j + \frac{1}{\eta} \partial_a \left(\sqrt{E} G_k^{rot} E_k^a \right) \approx 0 \end{aligned} \quad (15)$$

Evaluating the Dirac brackets of the rotation constraints G_i^{rot} with the basic fields, we find that they are the generators of the SU(2) gauge transformations:

$$\begin{aligned} [G_i^{rot}(x), \hat{E}_j^a(y)]_D &= \epsilon^{ijk} \hat{E}_k^a \delta^{(3)}(x, y), \\ [G_i^{rot}(x), A_a^j(y)]_D &= -\eta (\delta^{ij} \partial_a + \eta^{-1} \epsilon^{ikj} A_a^k) \delta^{(3)}(x, y). \end{aligned} \quad (16)$$

Thus, we have a $SU(2)$ gauge theory of gravity with all three topological parameters. The Barbero-Immirzi parameter η^{-1} acts as the coupling constant of gauge field A_a^i whereas the other two parameters θ, ϕ enter in the definition of its conjugate \hat{E}_i^a . Since the topological densities are all functions of the geometric fields (i.e. tetrads and spin-connections), addition of matter coupling does not affect such a gauge theoretic interpretation of gravity.

IV. CONCLUDING REMARKS

It is important to investigate the imports of these topological terms in the quantum theory of gravity. Although the Barbero-Immirzi parameter is known to appear in the area spectrum in Loop Quantum Gravity, the role of the other two parameters in quantum geometry is yet to be understood. Also, the question that whether these terms imply non-trivial topological sectors and potential instanton effects in the quantum theory similar to the non-abelian gauge theories demands a detailed study (for a relevant discussion, see [7] and the references within).

As a final remark, we note that the Dirac bracket between A_a^i and \hat{E}_i^a is not a canonical one, unlike the Barbero-Immirzi formulation. Although this need not be an issue as far as the classical theory is concerned, quantization based on these canonical variables is not straightforward. However, as demonstrated in [2], it is possible to find a suitable canonical pair which leads to the standard bracket, thus providing a smooth passage towards the quantum theory.

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